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ANALYTICAL SOLUTION OF THE EQUATIONS FOR A COAXIAL  
PLASMA GUN OPERATING IN THE SNOWPLOW MODE & WEAK  
COUPLING LIMIT(1)

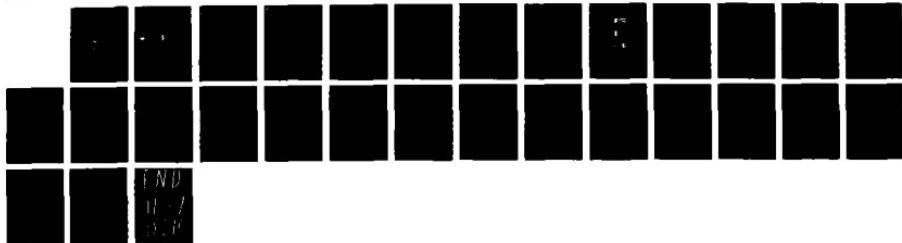
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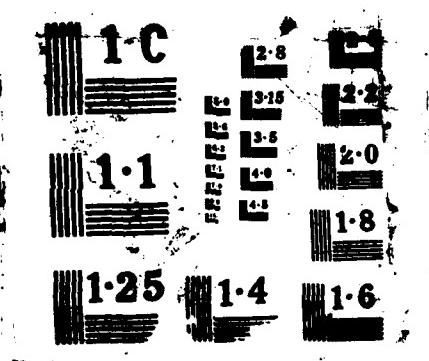
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# ANALYTICAL SOLUTION OF THE EQUATIONS FOR A COAXIAL PLASMA GUN OPERATING IN THE SNOWPLOW MODE.I. WEAK COUPLING LIMIT

Dr David Dietz

June 1987

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AIR FORCE WEAPONS LABORATORY  
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SNOWPLOW MODE.I. WEAK COUPLING LIMIT

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## I. INTRODUCTION

This technical report presents the analytical solution of the system of coupled ordinary differential equations (ODEs) which describes the time evolution of an ideal (i.e., zero resistance) coaxial plasma gun operating in the so-called snowplow mode (Refs. 1 and 2) being driven by a single loop RLC circuit. The full equations appear to be difficult to solve analytically in closed form although others (Refs. 3-13) have obtained approximate, asymptotic and numerical solutions to several variants of the system of equations. However, in the special situation where the effect of the coaxial gun on the circuit is negligible (i.e., the gun is fully influenced by the circuit but the circuit is negligibly influenced by the gun)--a state-of-affairs called the weak coupling limit--the system of coupled ODEs is readily solvable in closed form. This solution to the weakly coupled problem is presented here, leaving for a subsequent report the details of the solution to the fully coupled equations.

Section II presents the physical model of the system and introduces notation, and Section III describes the equations governing the system. Section IV presents the solution in the weak coupling limit, while Section V discusses the validity of the weak coupling limit. Finally, Section VI discusses relevant numerical results.

## II. THE PHYSICAL MODEL

A schematic diagram of the physical system under consideration is shown in Figure 1. The operation of this system is as follows.

Initially (time =  $0^-$ , Fig. 1a), the capacitor, with capacitance  $C$ , is fully charged, with the charge on the positive plate being  $q(0^-) = Q > 0$ ; the circuit, with inherent inductance  $L_0$  and inherent resistance  $R_0$ , is open; the current is  $i(0^-) = 0$ ; and the region between the straight coaxial cylinders, with inner and outer radii  $r_i$  and  $r_o$ , is full of unionized gas of uniform mass density,  $\rho$ .

Then (Fig. 1b), the switch is closed and the potential difference across the inner and outer electrodes causes the gas in a thin washer-like layer next to the rear insulator to break down, forming a conducting plasma sheath there and closing the circuit, allowing a current to flow and the capacitor to begin discharge. Call the first instant time zero, when essentially all the gas in the thin layer is in a plasma state (an idealization), and take

$$q(0) = Q \text{ and } i(0) = 0 = q(0) \quad (1)$$

Also, denote the thickness of the initial plasma sheath by  $\Delta$ ,  $\Delta > 0$ .

In the snowplow mode of operation, the current through this initial plasma sheath rapidly heats it up, resulting in the formation of a shock wave that begins to propagate down the gun. This shock wave initially deposits some of its energy in a thin layer of unionized gas directly in front of the initial plasma sheath, causing this gas to ionize and become heated plasma also. Simultaneously, the radial current through the plasma interacts with the azimuthal magnetic field produced by the coaxial current to produce an axial force on the now augmented plasma mass, causing it to begin to accelerate down the gun. (See Ref. 2 for a more detailed description.) The net result of these two effects is that the total mass of plasma in the gun is now greater than that in the original plasma sheath and, furthermore, this more massive plasma sheath is moving down the gun. This process is repeated

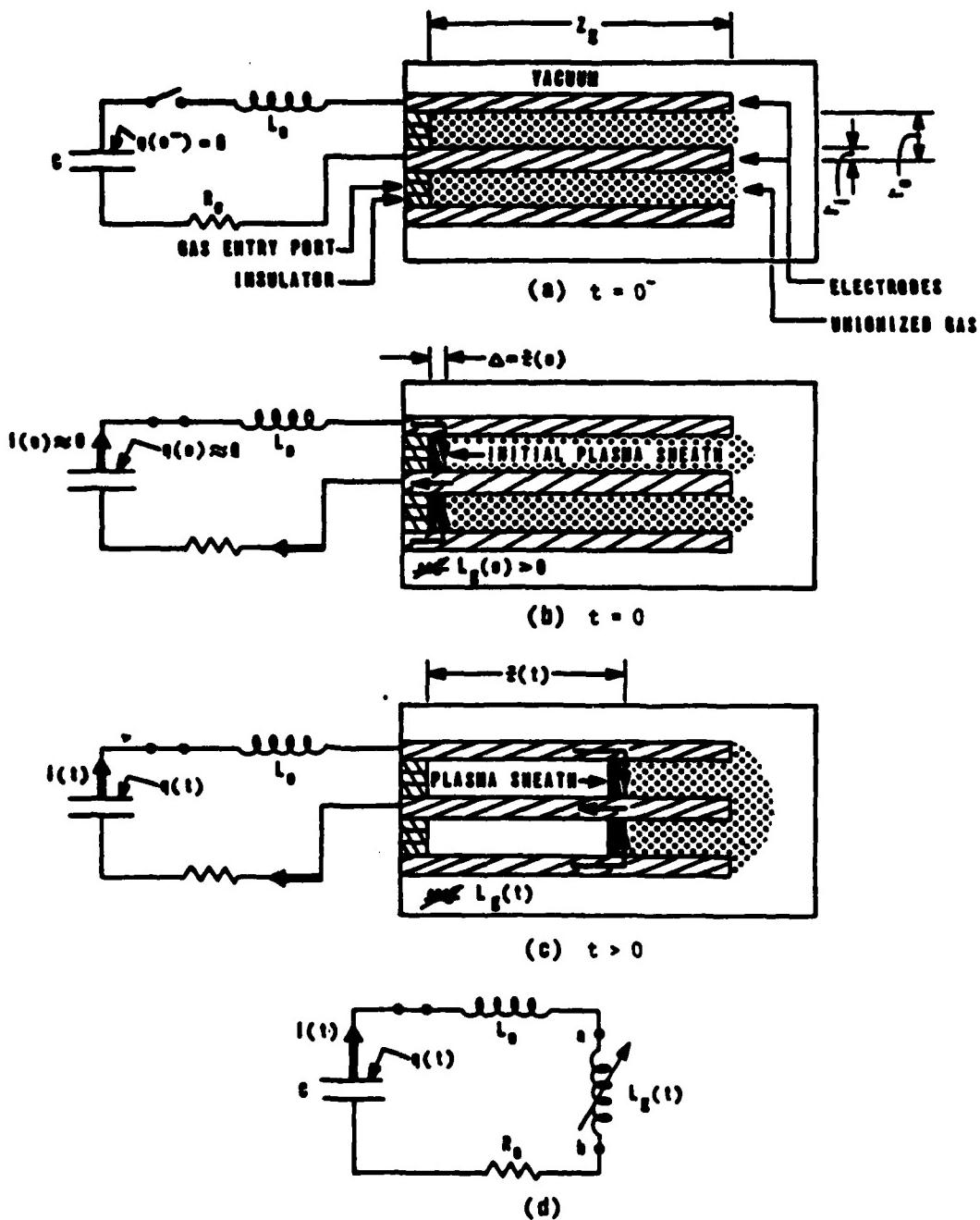


Figure 1. The system at various times and the equivalent circuit. (The dark arrows in (b) and (c) indicate the current flow.)

continually (with the initial plasma sheath being replaced by the updated plasma sheath), and at any time  $t > 0$ , an accreting mass of plasma moves down the gun with instantaneous mass equal to that of the unionized gas initially (i.e., at  $t = 0$ ) contained between the insulator and the current position of the leading edge (assumed sharp, flat, and perpendicular to the gun axis of symmetry) (Fig. 1c). Denote the distance from the leading edge of the plasma sheath to the front of the rear insulator by  $z(t)$ . Note that

$$z(0) = \Delta \quad (2a)$$

Also, take the velocity,  $\dot{z}(t)$ , of the sheath to be zero initially:

$$\dot{z}(0) = 0 \quad (2b)$$

This definition was chosen for  $z$  because we want the mass,  $m(t)$ , of the plasma sheath to be proportional to  $z(t)$ .

Also note that our definitions of the circuit quantities imply

$$\bullet \quad i(t) = -dq/dt \quad (3)$$

### III. THE GOVERNING EQUATIONS

We now derive the two equations governing the time evolution of the system: the circuit equation and the equation of motion of the plasma sheath.

#### 1. THE CIRCUIT EQUATION

The circuit is a single loop RLC circuit with capacitance  $C$  (other [stray] capacitance is neglected); with inductance

$$L(t) = L_0 + L_g(t) \quad (4)$$

where  $L_g(t)$  is the time-dependent inductance of the portion of the gun between the front of the rear insulator and the leading edge of the plasma sheath:

$$L_g(t) = [(\mu_0/2\pi)\ln(r_0/r_f)]z(t) \quad (\text{MKS}) \quad (5)$$

[note that  $L_g(0) = \Delta(\mu_0/2\pi)\ln(r_0/r_f) > 0$ ; and with resistance  $R(t) = R_0 + R_w(t) + R_p(t)$ , where  $R_w(t)$  is the resistance contributed by the portion of the gun walls lying between the front of the rear insulator and the leading edge of the plasma sheath (the resistance of the portion of the walls bounding the rear insulator is included in  $R_0$ ), and  $R_p(t)$  is the resistance contributed by the plasma. Assume here that the gun is ideal, i.e.,  $R_w(t) = 0$ , and that the plasma has infinite conductivity, so that  $R_p(t) = 0$ ; hence,  $R(t) = R_0$ . (Alternatively, one may assume  $R_p(t)$  is some nonzero constant which has been absorbed into  $R_0$ .)]

The circuit equation is standard except for the presence of a variable inductance. Note how this variable inductance is handled in both the Kirchhoff Law treatment and the energy conservation treatment of the circuit (Fig. 1d).

a. Kirchhoff Law

We have

$$(q/c) - R_0 i - L_0 (di/dt) + (V_b - V_a) = 0$$

where  $V_b - V_a$  is the induced EMF,  $\xi$ , across  $L_g$ . But

$$V_b - V_a = \xi = -d\phi(t)/dt = -d[L_g(t)i]/dt = -L_g(t)(di/dt) - i(dL_g/dt) \quad (6)$$

where  $\phi(t)$  is the flux linking the variable inductor, and the inductance  $L_g(t)$  is defined by

$$\phi(t) = L_g(t)i(t) \quad (7)$$

So

$$(q/c) - [R_0 + (dL_g/dt)]i - [L_0 + L_g(t)](di/dt) = 0$$

Now recalling that  $i = -\dot{q}$  (Eq. 3),

$$[L_0 + L_g(t)]\ddot{q} + [R_0 + L_g(t)]\dot{q} + (1/C)q = 0 \quad (8)$$

which is the desired circuit equation. Note that when  $L_g > 0$ ,  $L_g$  is equivalent to a resistance, and the  $iL_g$  term is in some sense a dissipative one; this is made more precise in the energy conservation treatment.

b. Energy Conservation

To use this approach, one needs expressions for the amount of energy stored in the capacitor and in the inductor at any time  $t$ . For an ideal capacitor with constant capacitance  $C$ , this is considered to be the amount of energy invested in bringing the capacitor to charge  $q(t)$  from charge zero, namely,  $(1/2C)q^2(t)$ ; and for an ideal inductor with constant inductance  $L_0$ , this is

considered to be the amount of energy invested in bringing the inductor to current  $i(t)$  from current zero, namely  $(1/2)L_0 i^2(t)$ .

For an ideal variable inductor, the energy stored is also considered to be the energy invested and is calculated as follows: Let  $W(t)$  denote the amount of energy invested in "currenting" the variable inductor, with inductance  $L_g(t)$ , from time zero to time  $t$  and let  $\xi(t)$  denote the induced EMF across this inductor at time  $t$ . Then

$$\frac{dW}{dt} = |\xi| i = L_g(t)i(di/dt) + i^2(dL_g/dt)$$

where Eq. 6 has been used to get the second equality.  
Hence

$$dW = d\left[\frac{1}{2}L_g(t)i^2\right] + \frac{1}{2}i^2\left(dL_g/dt\right)$$

and

$$W(t) = \frac{1}{2}L_g(t)i^2(t) + \frac{1}{2}\int_0^t i^2(t')L_g(t')dt' \quad (9)$$

Conservation of energy for the circuit now gives

$$\frac{d}{dt}\left[(1/2C)q^2 + \frac{1}{2}L_0 i^2 + \frac{1}{2}L_g(t)^2 + \frac{1}{2}\int_0^t i^2(t')L_g(t')dt'\right] = -i^2R_0$$

or

$$\frac{d}{dt}\left[(1/2C)q^2 + \frac{1}{2}L(t)i^2\right] = -i^2R_0 - \frac{1}{2}i^2L \quad (10)$$

where we have used Eq. 4. Now using  $i = -\dot{q}$  in Eq. 10 yields Eq. 8.

According to Eqs. 9 and 10, one may interpret  $(1/2)L(t)i^2$  as the amount of energy stored in the inductor at time  $t$  and further available as electromagnetic energy to drive the circuit, and  $(1/2)\int_0^t i^2(t')L(t')dt'$  as the amount of energy stored in the inductor at time  $t$  and not further available as electromagnetic energy to drive the circuit, with  $(1/2)i^2L$  being the rate of

"dissipation" of circuit electromagnetic energy by the variable inductor. Of course, this is not dissipation in the usual sense because the lost electromagnetic energy all appears as (mechanical) kinetic energy of the deforming variable inductor; in our case, it appears as kinetic energy of the plasma sheath.

## 2. EQUATION OF MOTION OF THE PLASMA SHEATH

The motion of the plasma sheath is that of an accreting mass, with instantaneous momentum  $m(t)\dot{z}(t)$ , being pushed down the gun by the force associated with the energy "dissipated" in the variable inductor. By our definition of  $z(t)$ , the mass is given by

$$m(t) = \rho\pi\left(r_0^2 - r_i^2\right)z(t)$$

where  $\rho$  is the density of the unionized gas. The model makes no precise statement about the thickness of the plasma sheath and, hence, none about its density.

An expression for the force on the plasma sheath may be derived using either of the two points of view following:

a. All the energy dissipated in the variable inductor is transformed into translational kinetic energy of the sheath; that is, the sheath is considered to be mechanically rigid. (For this reason, the sheath is often called a slug.) If  $F$  denotes the force on the sheath, then  $(1/2)i^2L_g = F\cdot\dot{z}(t)$ . Now by Eq. 5,

$$\dot{L}_g = [(\mu_0/2\pi)\ln(r_0/r_i)]\dot{z}(t)$$

so whenever  $\dot{z}(t) \neq 0$

$$F = [(\mu_0/4\pi)\ln(r_0/r_i)]\dot{q}^2 \quad (11)$$

b. All of the energy dissipated in the variable inductor is expended in the work done by the magnetic field pressure in pushing the sheath down the gun; that is, there is no field diffusion into the plasma. This is consistent with the fact that we have taken  $R_p = 0$ . The pressure on the rear of the sheath at radius  $r$  is  $B^2(r)/2\mu_0$ , where  $B(r) = \mu_0 i/2\pi r$ , so that

$$F = (2\pi/2\mu_0) \int_{r_i}^{r_o} B^2(r) r dr = [(\mu_0/4\pi) \ln(r_o/r_i)] \dot{q}^2$$

in agreement with Eq. 11.

The equation of motion is then

$$\rho\pi \left( r_o^2 - r_i^2 \right) \frac{d}{dt} (z\ddot{z}) = (\mu_0/4\pi) \ln(r_o/r_i) \dot{q}^2$$

or

$$z\ddot{z} + \dot{z}^2 = \left[ \left( \mu_0/4\pi^2 \rho \right) \ln(r_o/r_i) / \left( r_o^2 - r_i^2 \right) \right] \dot{q}^2 \quad (12)$$

In summary, the governing equations for our physical system are (Eqs. 1, 2a, 2b, 5, 8 and 12):

$$[L_o + bz(t)]\ddot{q} + [R_o + b\dot{z}(t)]\dot{q} + (1/C)q = 0 \quad (13)$$

$$z\ddot{z} + \dot{z}^2 = a\dot{q}^2(t) \quad (14)$$

$$q(0) = Q > 0 \quad \dot{q}(0) = 0 \quad z(0) = \Delta > 0 \quad \dot{z}(0) = 0 \quad (15)$$

where

$$a = (\mu_0/4\pi^2 \rho) \ln(r_o/r_i) / (r_o^2 - r_i^2) \quad (16)$$

and

$$b = (\mu_0/2\pi) \ln(r_o/r_i) \quad (17)$$

We also require that

$$z(t) > 0 \text{ for all } t > 0 \quad (18)$$

The parameters  $a$  and  $b$  may be thought of as coupling parameters:  $A$  is a measure of the influence of the circuit on the gun and  $b$  is a measure of the influence of the gun on the circuit. Our weak coupling limit is (approximately) the one in which the parameter  $b$  is considered to be small. This should not be confused with the small coupling limit (Ref. 8) in which the parameter  $a$  is considered to be small.

## IV. SOLUTION IN THE WEAK COUPLING LIMIT

The details of the solution of the governing equations in the weak coupling (WC) limit follow. In the next section, the conditions are examined under which the WC limit is applicable.

Given  $\dot{q}(t)$ , Eq. 14 is easily solved for arbitrary initial conditions by noting that it may be written as

$$\frac{d^2(z^2)}{dt^2} = a\dot{q}^2$$

Then

$$z(t) = \pm [z(0)^2 + 2z(0)\dot{z}(0)t + 2a \int_0^t \int_0^v \dot{q}^2(u) du dv]^{\frac{1}{2}} \quad (19)$$

For our conditions we have

$$z(t) = \Delta [1 + (2a/\Delta^2) \int_0^t \int_0^v \dot{q}^2(u) du dv]^{\frac{1}{2}} \quad (20)$$

Of course,  $\dot{q}$  depends upon  $z$  via Eq. 13 so  $\dot{q}(t)$  is not in fact "given" in general. However, in the special case that

$$bz(t) \ll L_0 \quad (21a)$$

and

$$b\dot{z}(t) \ll R_0 \quad (21b)$$

for all times,  $t$ , of interest, the circuit equation becomes

$$L_0 \ddot{q} + R_0 \dot{q} + (1/C)q = 0 \quad (22)$$

which equation is easily solvable for  $\dot{q}(t)$ ; insertion of this  $\dot{q}(t)$  into Eq. 20 then yields the time evolution,  $z(t)$ , of the plasma sheath. Equations 21 and 22 give the precise definition of the WC limit.

Concerning the solution for  $z(t)$ , note from Eq. 20 that

$$\dot{z}(t) = (a/z) \int_0^t \dot{q}^2(u) du \quad (23)$$

so that

$$\dot{z}(t) > 0 \text{ for all } t > 0 \quad (24)$$

Also, note from Eq. 20 that

$$z(t) \geq \Delta > 0 \text{ for all } t \geq 0 \quad (25)$$

Hence, the plasma sheath is always moving down the gun (and never moving backwards up the gun). Since the gun has finite length, the plasma sheath eventually reaches the muzzle end, say at time  $= t_g$ . The times of interest referred to after Eqs. 21 are then  $0 \leq t \leq t_g$ . The model has no applicability for  $t > t_g$ .

The solution to the WC limit circuit equation (Eq. 22), with initial conditions on  $q$  and  $\dot{q}$  given by Eq. 15, is well known to be, in the non-critically damped case,

$$q(t) = Q \exp(-R_0 t / 2L_0) [\cos \Omega t + (R_0 / 2\Omega L_0) \sin \Omega t] \quad (26)$$

where

$$\Omega = [(1/L_0 C) - (R_0^2 / 4L_0^2)]^{1/2} \neq 0$$

and where, if  $\Omega = i\omega$  with  $\omega$  real, then  $\cos \Omega t = \cos(i\omega t) = \cosh \omega t$  and  $\sin \Omega t = \sin(i\omega t) = i \sinh \omega t$ . In the critically damped case,

$$q(t) = Q \exp(-R_0 t / 2L_0) [1 + (R_0 / 2L_0)t] \quad (27)$$

Of course, the general initial value problem with  $\dot{q}(0)$  arbitrary could be solved easily, but for notational simplicity later on [aside from the issue

of the natural initial conditions for our physical model], we choose not to do so: the solution for  $z(t)$  with  $\dot{q}(0) = 0$  will already be notationally cumbersome. The same comment applies to the initial value problem for  $z(t)$ : we use Eq. 20 rather than Eq. 19 to find  $z(t)$  from  $q(t)$ . Substituting Eq. 26 or 27 into Eq. 20, one gets, respectively (using  $v = R_0/2L_0$  and  $\delta = R_0/2\Omega L_0$ ),

$$\begin{aligned} -z(t) = \Delta & \left\{ 1 + \left( \frac{\alpha Q^2}{2\Delta^2} \right) \left( e^{-2\Delta t} \left[ \frac{1}{2} (1 - \delta^2) \cos 2\Delta t + \delta \sin 2\Delta t \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} [\delta^2 + (1/\delta^2)] + 1 \right] + \left[ \delta + (1/\delta) \right] \Delta t - \frac{1}{2} [3 + (1/\delta^2)] \right) \right\}^{1/2} \end{aligned} \quad (28)$$

and

$$z(t) = \Delta \left\{ 1 + \left( \frac{\alpha Q^2}{2\Delta^2} \right) \left( e^{-2\Delta t} \left[ (\nu t)^2 + 2\nu t + \frac{3}{2} \right] + \nu t - \frac{3}{2} \right) \right\}^{1/2} \quad (29)$$

These are the desired solutions for the motion of the plasma sheath in the WC limit.

For completeness, we also present expressions for the velocity,  $\dot{z}(t)$ , in the two cases. They are

$$\dot{z}(t) = (\Delta)(\Delta/z)(\alpha Q^2/4\Delta^2)(1+\delta^2) \left\{ e^{-2\Delta t} \left[ \delta \cos 2\Delta t - \sin 2\Delta t - [\delta + (1/\delta)] \right] + (1/\delta) \right\}^{1/2} \quad (30)$$

and

$$\dot{z}(t) = (\nu \Delta)(\Delta/z)(\alpha Q^2/4\Delta^2) \left\{ 1 - e^{-2\Delta t} \left[ 2(\nu t)^2 + 2\nu t + 1 \right] \right\}^{1/2} \quad (31)$$

Note that the issue of the qualitative dependence of the solution of the circuit equation (Eq. 13) upon the coefficients of that equation has been ignored. That is, for any fixed time  $t$  the nature of the solution to Eq. 13 depends upon the sign of

$$D(t) = [R_0^2 - (4L_0/C)] + bA(t)$$

where

$$A(t) = b\dot{z}^2(t) + 2R_0\dot{z}(t) - (4/C)z(t)$$

Using Eqs. 13-15 it may be shown that  $A(0) = -4\Delta/C$ ,  $\dot{A}(0) = 0 = \ddot{A}(0)$ , and  $\ddot{A}(0) = 4aR_0Q^2/\Delta C^2(L_0 + b\Delta)^2 > 0$  so that  $A(t)$  is not constant in time. One may therefore envision a choice of  $R_0$ ,  $L_0$ ,  $C$  such that  $D(t)$  changes sign at least once during  $[0, t_g]$ , and so speculate that the nature of the damping changes during the motion of the sheath.

## V. VALIDITY OF THE WEAK COUPLING LIMIT

In this section some conditions are examined under which the WC approximation is applicable. The length of the coaxial gun gas compartment is denoted by  $Z_g$ .

First, consider the requirement imposed by Eq. 21a. An upper bound on  $bz(t)$  for  $0 \leq t \leq t_g$  is  $bZ_g$ . If we take  $x \ll y$  to mean  $\alpha x \leq y$  (usually  $\alpha \geq 10$ ), then it is necessary and sufficient that

$$L_0 \geq abZ_g = \alpha(\mu_0/2\pi)\ln(r_0/r_1)Z_g \quad (32)$$

Next, consider the requirement imposed by Eq. 21b. An upper bound on  $b\dot{z}(t)$  for  $0 \leq t \leq t_g$  is not as readily obtainable as that on  $bz(t)$ ; the result is

$$b\dot{z}(t) \leq ba^{\frac{1}{2}}Q/(L_0C)^{\frac{1}{2}}, \quad 0 \leq t \leq t_g \quad (33)$$

which we will demonstrate shortly. It is then sufficient that

$$R_0 \geq aba^{\frac{1}{2}}Q/(L_0C)^{\frac{1}{2}} \quad (34)$$

If  $V_0$  is the initial potential difference across the capacitor, then  $Q = CV_0$  and so Eq. 34 becomes

$$R_0 \geq aba^{\frac{1}{2}}V_0(C/L_0)^{\frac{1}{2}}$$

Combining this with Eq. 32, the condition on  $R_0$  then becomes

$$\begin{aligned} R_0 &\geq (aabV_0^2C/Z_g)^{\frac{1}{2}} \\ &= a^{\frac{1}{2}}\left(\mu_0^2/8\pi^2\right)^{\frac{1}{2}} \left(V_0^2C\right)^{\frac{1}{2}} \ln(r_0/r_1)/\left[\rho Z_g(r_0^2 - r_1^2)\right]^{\frac{1}{2}} \end{aligned} \quad (35)$$

Noting that  $\mu_0 = 4\pi \times 10^{-7}$  H/m (MKS), we get sufficient conditions on  $L_0$  and  $R_0$ , in terms of the other system parameters, for the WC limit to be valid:

$$L_0 \geq 0.2aZ_g \ln(r_0/r_i) \quad (\mu H) \quad (36)$$

and

$$R_0 \geq 7.979 \times 10^{-3} a^{\frac{1}{2}} \left( V_0^2 C \right)^{\frac{1}{2}} \ln(r_0/r_i) / \left[ \rho Z_g r_i^2 \left( (r_0/r_i)^2 - 1 \right) \right]^{\frac{1}{2}} \quad (\text{m}) \quad (37)$$

We now prove the result in Eq. 33. First, it follows from Eq. 10 (energy conservation) and the condition  $L > 0$  that, for  $0 \leq t \leq t_g$ , we have

$$\frac{1}{2} L_0 i^2(t) \leq \frac{1}{2} L(t) i^2(t) \leq (1/2C) Q^2$$

so that

$$\dot{q}^2(t) \leq Q^2/L_0 C, \quad 0 \leq t \leq t_g \quad (38)$$

Hence, from Eqs. 23-25, for  $0 \leq t \leq t_g$ , we have

$$\dot{z}(t) \leq [a/z(t)] \left( Q^2/L_0 C \right) t_g \leq (a/\Delta) \left( Q^2/L_0 C \right) t_g$$

Since  $\ddot{z}$  is bounded above on compact  $[0, t_g]$ , it attains a global maximum there. This maximum is attained at an interior stationary point, where  $\ddot{z} = 0$ , or at the endpoint  $t = t_g$ , where  $\ddot{z} \geq 0$  [for if  $\ddot{z}(t_g) < 0$  then  $\ddot{z}(t_g^-) > \ddot{z}(t_g)$  which contradicts the fact that  $\ddot{z}(t_g)$  is a global maximum]. In any case, if we denote by  $t_{\max}$  a time at which  $\ddot{z}$  achieves its global maximum, and denote  $\ddot{z}(t_{\max})$  by  $\ddot{z}_{\max}$ , then  $\ddot{z}(t_{\max}) \geq 0$  and Eqs. 14 and 18 imply that

$$\dot{z}_{\max}^2 \leq a \left[ \dot{q}(t_{\max}) \right]^2 \quad (39)$$

Now using Eq. 38 in Eq. 39 gives

$$\dot{z}_{\max}^2 \leq a Q^2 / L_0 C$$

whereupon Eq. 33 follows immediately.

## VI. NUMERICAL RESULTS

In this final section, numbers and graphs relevant to the preceding theory are presented. We arbitrarily choose  $\alpha = 10$ ; upon absorption of  $\alpha^{\frac{1}{2}}$ , the constant in Eq. 37 becomes  $2.523 \times 10^{-4}$ .

Tables 1 and 2 give the minimum values of  $L_0$  required by Eq. 36 and of  $R_0$  required by Eq. 37, respectively, for various representative values of  $r_0/r_i$  and  $Z_g$ . In the latter case,  $V_0$ ,  $r_i$  and  $C/\rho$  are fixed at the typical values  $5 \times 10^6$  V, 0.03 m and  $10^{-2}$  F/kg·m<sup>3</sup> respectively; this last choice is consistent with the range of values of  $C$  and  $\rho$  representative of plasma guns operating in the snowplow mode, namely,  $10^{-6} \leq C \leq 10^{-3}$  (F) and  $10^{-5} \leq \rho \leq 10^{-3}$  (kg/m<sup>3</sup>).

Figures 2-4 show  $i(t)$ ,  $z(t)$  and  $\dot{z}(t)$  for representative underdamped, critically damped and overdamped cases, respectively. For these graphs we use the dimensionless forms

$$i/\Omega Q = (1 + \delta^2) e^{-\delta \Omega t} \sin \Omega t$$

from Eq. 26,  $z/\Delta$  from Eq. 28 and  $\dot{z}/\Omega \Delta$  from Eq. 30 for the noncritically damped case, and we use the dimensionless forms

$$i/vQ = vte^{-vt}$$

from Eq. 27,  $z/\Delta$  from Eq. 29 and  $\dot{z}/v\Delta$  from Eq. 31 for the critically damped case.

Concerning the choices of values of  $\delta$  and dimensionless coupling strength (see Eqs. 28-31)

$$\kappa \equiv \Omega Q^2 / \Delta^2$$

note the following:

TABLE 1. MINIMUM VALUES OF  $L_0$  ( $\mu\text{H}$ ) FOR WHICH  
THE WC LIMIT IS VALID FOR VARIOUS  
VALUES OF  $Z_g$  (m) AND  $r_o/r_i$ ;  $\alpha = 10$

$Z_g \backslash r_o/r_i$	1.1	2	10
0.1	0.019	0.139	0.461
0.5	0.096	0.695	2.31
1	0.191	1.39	4.61

TABLE 2. MINIMUM VALUES OF  $R_o$  ( $\text{m}\Omega$ )<sup>a</sup> FOR WHICH  
THE WC LIMIT IS VALID FOR VARIOUS  
VALUES OF  $Z_g$  (m) AND  $r_o/r_i$ ;  $C/\rho = 10^{-2}$   
 $F/\text{kg} \cdot \text{m}^3$ ,  $V_o = 5 \times 10^4$  V,  $r_i = 0.03$  m;  
 $\alpha = 10$

$Z_g \backslash r_o/r_i$	1.1	2	10
0.1	27.7	53.2	30.8
0.5	12.3	23.7	13.7
1	8.73	16.8	9.73

<sup>a</sup>Values of  $L_0$  given in Table 1, namely  $L_0^{\min}$ , are assumed here. For values,  $L_0$ , of inductance greater than  $L_0^{\min}$ , the minimum values of  $R_o$  given here decrease as  $(L_0^{\min}/L_0)^{\frac{1}{2}}$  (see Eq. 34).

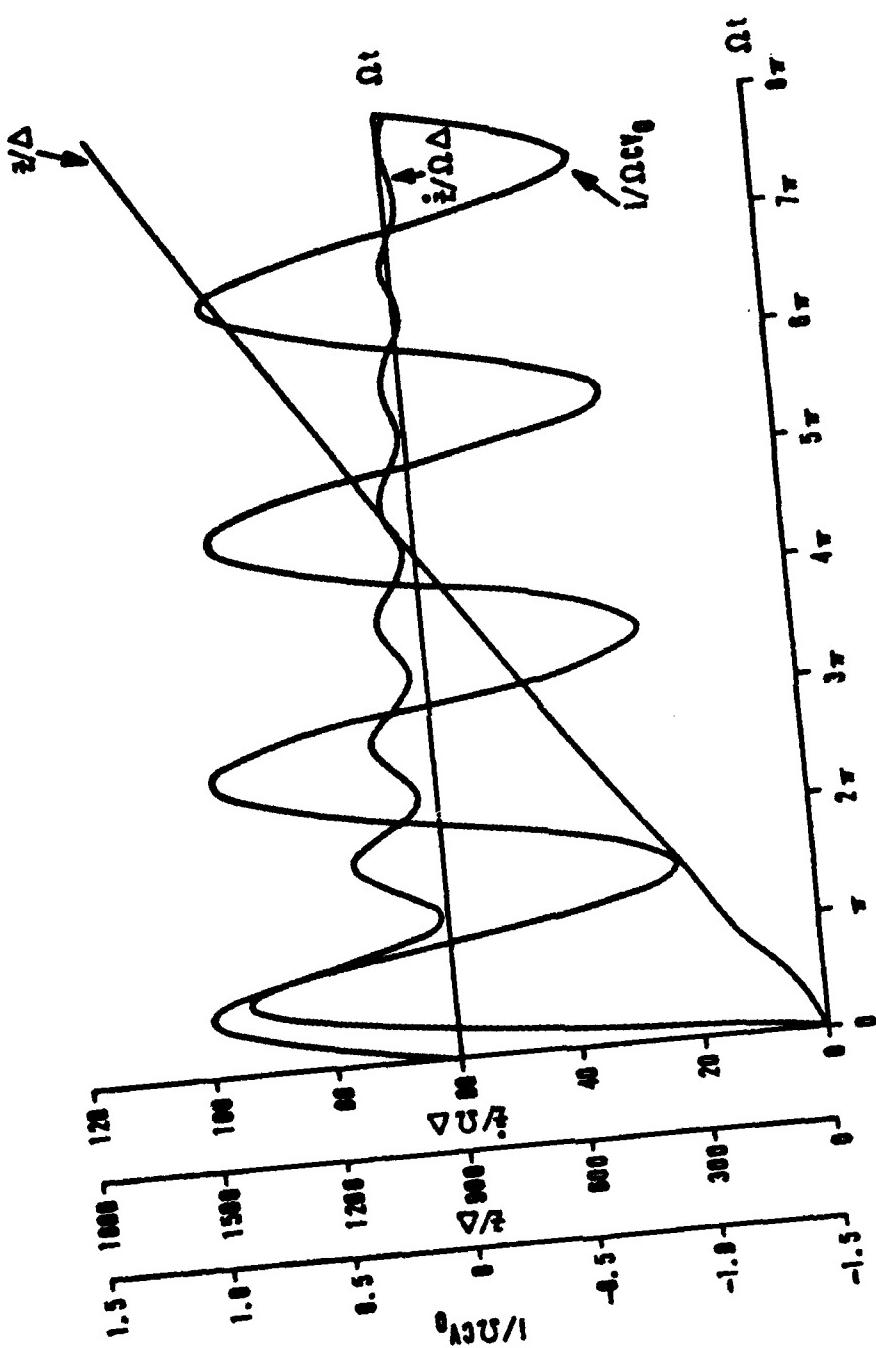


Figure 2. Dimensionless  $i(t)$ ,  $z(t)$ , and  $\dot{z}(t)$  for the underdamped case:  
 $\delta = 0.0119$  and  $\kappa = 10^3$ .

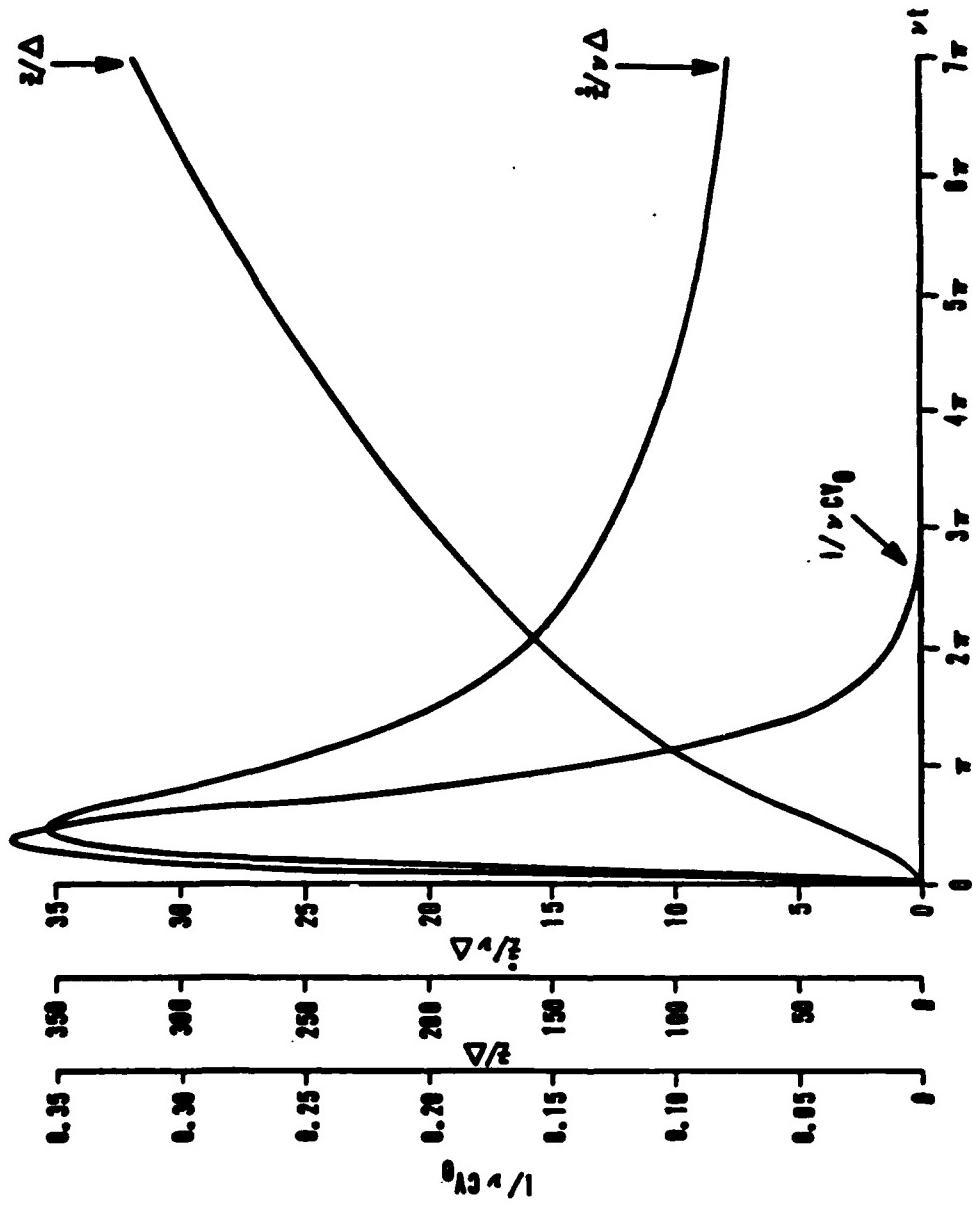


Figure 3. Dimensionless  $i(t)$ ,  $z(t)$ , and  $\dot{z}(t)$  for the critically damped case:  
 $\kappa = 10^3$ .

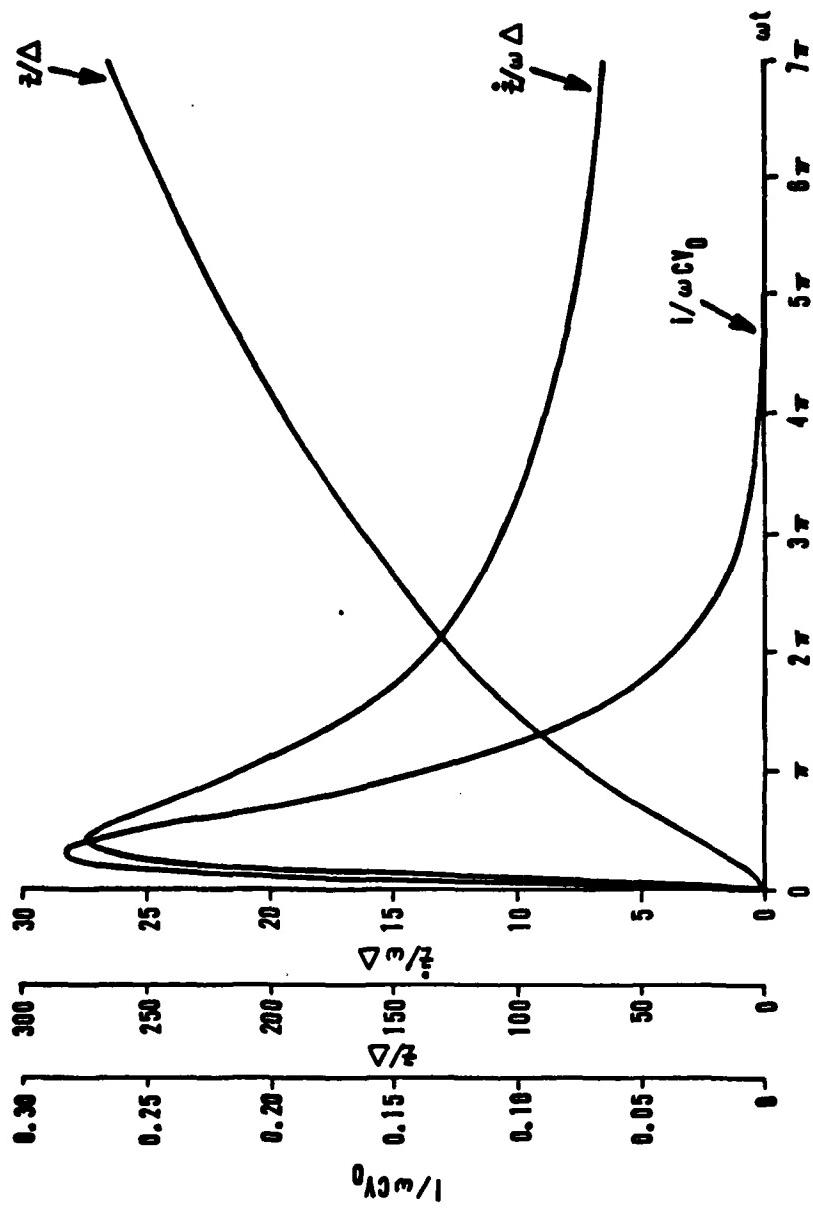


Figure 4. Dimensionless  $i(t)$ ,  $z(t)$  and  $\dot{z}(t)$  for the overdamped case:  
 $\delta = 1.407$  and  $\kappa = 10^3$ .

a. The choices  $\delta = 0.0119$  for the underdamped case and  $|\delta| = 1.407$  for the overdamped case (where by our convention,  $\delta$  is purely imaginary) correspond to  $L_0 = 335 \text{ nH}$ ,  $C = 3 \mu\text{F}$  and  $R_0 = 8 \text{ m}\Omega$  or  $R_0 = 950 \text{ m}\Omega$ , respectively (with  $\Omega = 1.003 \times 10^6 \text{ s}^{-1}$  or  $\omega = 1.008 \times 10^6 \text{ s}^{-1}$ , respectively). (For critical damping with this  $L_0$  and  $C$ ,  $R_0 = 668\frac{1}{3} \text{ m}\Omega$  and  $\nu = 0.997 \times 10^6 \text{ s}^{-1}$ .) Tables 1 and 2 show that these choices of  $L_0$  and  $R_0$  are sufficient for the validity of the WC limit under some circumstances, e.g., when  $r_0/r_i = 1.1$  and  $Z_g \lesssim 2 \text{ m}$  (and  $r_i = 0.03 \text{ m}$  and  $V_0 = 5 \times 10^4 \text{ V}$ ). These choices of  $R_0$ ,  $L_0$  and  $C$  also make  $\Omega$ ,  $\omega$ ,  $\nu \approx 1.0 \times 10^6 \text{ s}^{-1}$  so that the numerical values of  $\Omega t$ ,  $\omega t$  and  $\nu t$  on plot abscissas are very nearly equal to the numerical values of  $t$ , in units of microseconds, to which they correspond.

b. The possible sensible values of  $\kappa$  span a range of at least twelve orders of magnitude, from  $1.83 \times 10^{-7}/\Delta^2$  to  $1.45 \times 10^5/\Delta^2$ , corresponding to the two sets of values  $\{r_0/r_i = 10, r_i = 0.05 \text{ m}, C = 1 \mu\text{F}, \rho = 10^{-3} \text{ kg/m}^3, V_0 = 2.5 \times 10^3 \text{ V}\}$  and  $\{r_0/r_i = 1.1, r_i = 0.01 \text{ m}, C = 1000 \mu\text{F}, \rho = 10^{-5} \text{ kg/m}^3, V_0 = 10^5 \text{ V}\}$ , respectively. The actual value of  $\Delta$  is unimportant since, as can be seen from Eqs. 28-31, it just establishes a basic scale length in terms of which the quantities  $z$  and  $\dot{z}$  are measured. For definiteness we take  $\Delta = 10^{-3} \text{ m}$ . Then  $1 \times 10^{-1} \lesssim \kappa \lesssim 1 \times 10^{10}$ . Clearly, for small  $\kappa$ ,  $\dot{z}(t)$  is relatively small and  $z(t)$  is relatively flat. To exhibit some structure in the graphs we choose a large value of  $\kappa$ , namely  $\kappa = 1 \times 10^3$ ; this corresponds to the set of values  $r_0/r_i = 1.1, r_i = 0.03 \text{ m}, C = 3 \mu\text{F}, \rho = 3 \times 10^{-4} \text{ kg/m}^3$ , and  $V_0 = 5 \times 10^4 \text{ V}$ .

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